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ECS 455: Mobile Communications
Chapter 3: Call Blocking Probability

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In this note, we look at derivations of the Erlang B formula and the Engsett formula, both of which estimate the call blocking probability when trunking is used. To do this, we need to borrow some concepts from queueing theory. Moreover, some basic analysis of stochastic processes including Poisson processes and Markov chains is needed. For completeness, working knowledge on these processes is summarized here as well. However, we do assume that the readers are familiar with concepts from basic probability course.

Contents

In performance evaluation of cellular systems or telephone networks, ***Erlang B formula*** [1, p 23], to be revisited in Section 2, is a formula for estimating the call blocking probability for a cell (or a sector, if sectoring is used) which has m "trunked" channels and the amount of ("offered") traffic is A Erlang:

$$P_b = \frac{\frac{A^m}{m!}}{\sum_{i=0}^{m} \frac{A^i}{i!}}. \tag{1}$$

It is directly used to determine the probability P_b that call requests will be blocked by the system because all channels are currently used. The amount of traffic (A) can be found by the product of the total call request rate λ and the average call duration ($1/\mu$).

When we design a cellular system, the blocking probability P_b should be less than some pre-determined value. In which case, the function above can be used to suggest the minimum number of channels per cell (or sector). If we already know the number of channels per cell (or sector) of the system, the (inverse of this) function can also be used to determined how many users the system can support.

The Erlang B formula (1) is derived under the following M/M/m/m assumptions:

(a) Blocked calls cleared

- No queue for call requests.
- For every user who requests service, there is no setup time and the user is given immediate access to a channel if one is available.
- If no channels are available, the requesting call is blocked without access and has no further influence on the system.

(b) Call generation/arrival is determined by a ***Poisson process***.

- Arrivals of requests are memoryless.

(c) There are an infinite number of users (with finite overall request rate).

- The finite user results always predict a smaller likelihood of blocking. So, assuming infinite number of users provides a conservative estimate.

(d) The duration of the time that a user occupies a channel is ***exponentially*** distributed.

(e) The call duration times are independent and they are also independent from the call request process.

(f) There are m channels available in the trunking pool.

- For us, m = the number of channels for a cell or for a sector.

Some of these conditions are captured by Figure 1.

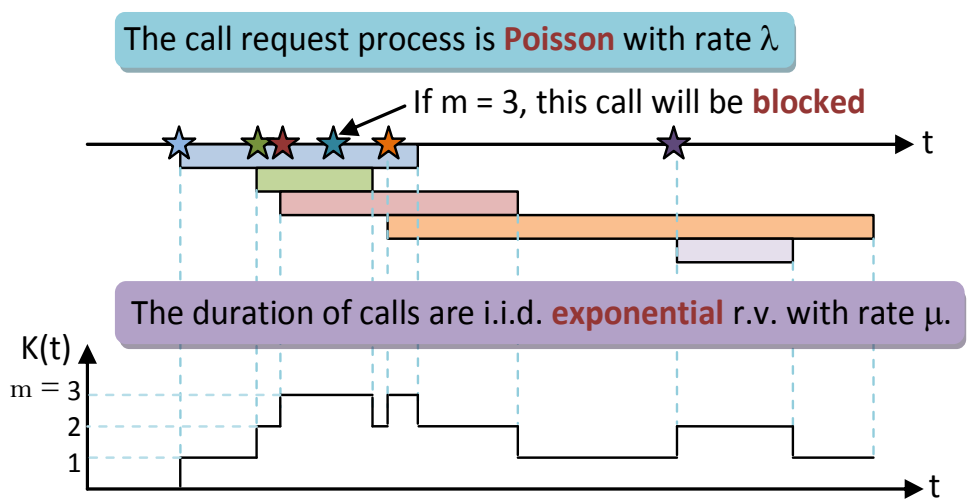


Figure 1: M/M/m/m Assumptions. Here, $m = 3$.

In this note, we will look more closely at these assumptions and see how they lead to the Erlang B formula. The goal is not limited to simply understanding of the derivation of the formula itself but, later on, we also want to try to develop a new formula that relaxes some of the assumptions above to make the analysis more realistic.

In Figure 1, we also show one new important parameter of the system: $K(t)$. This is the number of used channels at time t. When $K(t) < m$, new call request can be serviced. When $K(t) = m$, new call request(s) will be blocked. So, we can find the call blocking probability by looking at the value of $K(t)$. In particular, we want to find out the proportion of time the system has $K = m$. This key idea will be revisited again in Section 2.

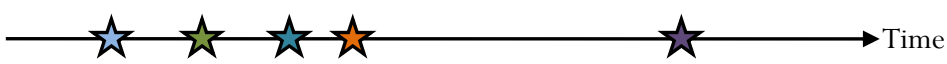
As seen in the assumptions, understanding of the Poisson process is important for the derivation of the Erlang B formula. Therefore, this process

and some probability concepts will be reviewed in Section 1. Most of the probability review will be put in footnotes so that they do not interfere with the flow of the presentation.

1 Poisson Processes

In this section, we consider an important random process called ***Poisson process*** (PP). This process is a popular model for customer arrivals or calls requested to telephone systems.

1.1. We start by picturing a Poisson Process as a random arrangement of "marks" (denoted by "x" or) on the time axis. These marks usually indicate the time that customers arrive in queueing models. In the language of "queueing theory", the marks denote ***arrival times***. For us, they indicate the time that call requests are made:



1.2. In this class, we will focus on one kind of Poisson process called ***homogeneous***[1] ***Poisson process***. Therefore, from now on, when we say "Poisson process", what we mean is "homogeneous Poisson process".

1.3. The first property that you should remember is that

there is only one parameter for Poisson process.

This parameter is the ***rate*** or ***intensity*** of arrivals (the average number of arrivals per unit time.) We use λ to denote[2] this parameter.

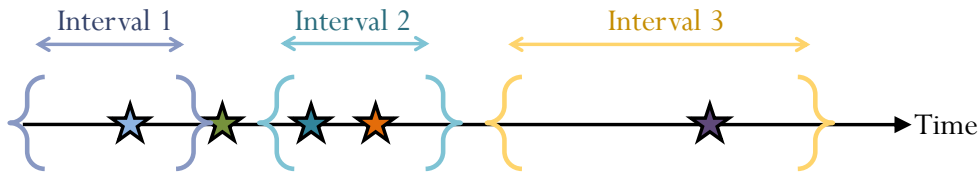
1.4. How can λ, which is the only parameter, controls Poisson process?

The key idea is that the Poisson process is as random/unstructured as a process can be.

[1]This is a special case of Poisson processes. More general Poisson process (which are called non-homogenous Poisson processes) would allow the rate to be time-dependent. Homogeneous Poisson processes have constant rates.

[2]For homogeneous Poisson process, λ is a constant. For non-homogeneous Poisson process, λ is a function of time, say $\lambda(t)$. Our λ is constant because we focus on homogeneous Poisson process

Therefore, if we consider many non-overlapping intervals on the time axis, say interval 1, interval 2, and interval 3 below,



and count the number of arrivals N_1, N_2 and N_3 in these intervals[3]. Then, the numbers N_1, N_2 and N_3 in our example above should be independent[4]; for example, knowing the value of N_1 does not tell us anything at all about what values of N_2 and N_3 will be. This is what we are going to take as a vague definition of the "complete randomness" of the Poisson process.

To summarize, now we have one more property of a Poisson process:

> The number of arrivals in non-overlapping intervals are independent.

[3]Note that the numbers N_1, N_2, and N_3 are random. Because they are counting the number of arrivals, we know that they can be any non-negative integers:

$$0, 1, 2, 3, \ldots.$$

Because we don't know their exact values, we describe them via the likelihood or probability that they will take one of these possible values. For example, for N_1, we describe it by

$$P[N_1 = 0], P[N_1 = 1], P[N_1 = 2], \ldots$$

where $P[N_1 = k]$ is the probability that N_1 takes the value k. Such list of numbers is a bit tedious. So, we define a function

$$p_{N_1}(k) = P[N_1 = k].$$

This function $p_{N_1}(\cdot)$ tells the probability that N_1 will take a particular value (k). We call p_{N_1} the probability mass function (pmf) of N_1. At this point, we don't know much about $p_{N_1}(k)$ except that its values will be between 0 and 1 and that

$$\sum_{k=0}^{\infty} p_{N_1}(k) = 1.$$

These two properties are the necessary and sufficient conditions for any pmf.

[4]By saying that something are independent, we mean it in terms of probability. In particular, when we say that N_1 and N_2 are independent, it means that

$$P[N_1 = k \text{ and } N_2 = m]$$

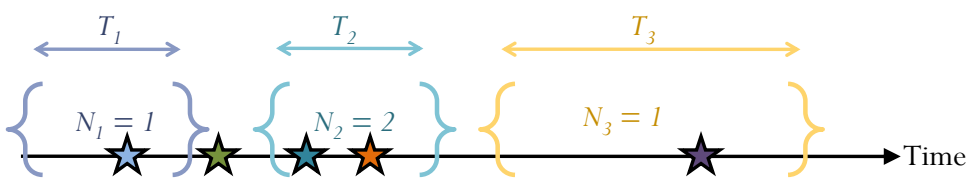
(which is the probability that $N_1 = k$ and $N_2 = m$) can be written as the product

$$p_{N_1}(k) \times p_{N_2}(k).$$

1.5. Do we know anything else about N_1, N_2, and N_3? Again, we have only one parameter λ for a Poisson process. So, can we relate λ with N_1, N_2, and N_3?

Recall that λ is the average number of arrivals per unit time. So, if $\lambda = 5$ arrivals/hour, then we expect that N_1, N_2, and N_3 should conform with this λ, statistically.

Let's first be more specific about the time durations of the intervals that we have earlier. Suppose their lengths are T_1, T_2, and T_3 respectively.



Then, you should expect[5] that

$$\begin{aligned} \mathbb{E}N_1 &= \lambda T_1, \\ \mathbb{E}N_2 &= \lambda T_2, \text{ and} \\ \mathbb{E}N_3 &= \lambda T_3. \end{aligned}$$

For example, suppose $\lambda = 5$ arrivals/hour and $T_1 = 2$ hour. Then you would see about $\lambda \times T_1 = 10$ arrivals during the first interval. Of course, the number of arrivals is random. So, this number 10 is an average or the expected number, not the actual value.

To summarize, we now know one more property of a Poisson process:

> For any interval of length T, the expected number of arrivals in this interval is given by
>
> $$\mathbb{E}N = \lambda T. \tag{2}$$

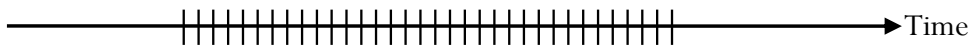
[5]Recall that $\mathbb{E}N_1$ is the expectation (average) of the random variable N_1. Here, the random variables are discrete. Therefore, formula-wise, we can calculate $\mathbb{E}N_1$ from

$$\mathbb{E}N_1 = \sum_{k=0}^{\infty} k \times P\left[N_1 = k\right];$$

that is the sum of the possible values of N_1 weighted by the corresponding probabilities

1.1 Discrete-time (small-slot) approximation of a Poisson process

1.6. The next key idea is to consider a small interval:
Imagine dividing a time interval of length T into n equal slots.



Then each slot would be a time interval of duration $\delta = T/n$. For example, if $T = 20$ hours and $n = 10,000$, then each slot would have length

$$\delta = \frac{T}{n} = \frac{20}{10,000} = 0.002 \text{ hour}.$$

Why do we consider small interval? The key idea is that as the interval becomes very small, then it is extremely unlikely that there will be more than 1 arrivals during this small amount of time. This statement becomes more accurate as we increase the value of n which decreases the length of each interval even further. What we are doing here is an approximation[6] of a continuous-time process by a discrete-time process.[7]

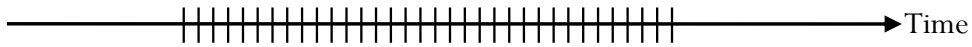
To summarize, we will consider the discrete-time approximation of the (continuous-time) Poisson process. In such approximation, the time axis is divided into many small time intervals (which we call "slots").

> When the interval is small enough, we can assume that at most 1 arrival occurs.

[6] You also do this when you plot a graph of any function $f(x)$. You divide the $x-$axis by many (equally spaced) values of x and then evaluate the values of the function at these values of x. You need to make sure that the values of x used are "dense" enough such that no surprising change in the function f is overlooked.

[7] If we want to be rigorous, we would have to bound the error from such approximation and show that the error disappear as $n \to \infty$. We will not do that here.

1.7. Let's look at the small slots more closely. Here, we let N_1 be the number of arrivals in slot 1, N_2 be the number of arrivals in slot 2, N_3 be the number of arrivals in slot 3, and so on as shown below.



Then, these N_i's are all Bernoulli random variables because they can only take the values 0 or 1. In which case, for their pmfs, we only need to specify one value $P[N_i = 1]$. Of course, knowing this, we can calculate $P[N_i = 0]$ by $P[N_i = 0] = 1 - P[N_i = 1]$.

Recall that the average $\mathbb{E}X$ of any Bernoulli random variable X is simply $P[X = 1]$.[8] So, if we know $\mathbb{E}X$ for Bernoulli random variable, then we know right away that $P[X = 1] = \mathbb{E}X$ and $P[X = 0] = 1 - \mathbb{E}X$.

Now, it's time to use what we learned about Poisson process. The slots that we consider before are of length $\delta = T/n$. So, the random variables $N_1, N_2, N_3, \ldots$ share the same expected value

$$\mathbb{E}N_1 = \mathbb{E}N_2 = \mathbb{E}N_3 = \cdots = \lambda\delta.$$

For example, with $\lambda = 5$, $T = 20$, and $n = 10,000$, the expected number of arrivals in a slot is $\lambda\delta = \lambda\frac{T}{n} = 0.01$ arrivals.

Because these N_i's are all Bernoulli random variables and because they share the same expected value, we can conclude that they are identically distributed; that is their pmf's are all the same. Furthermore, because the slots do not overlap, we also know that the N_i's are independent. Therefore,

> for small non-overlapping slots, the corresponding number of arrivals N_i's are i.i.d. Bernoulli random variables whose pmf's are given by
>
> $$p_1 = P[N_i = 1] = \lambda\delta \quad \text{and} \quad p_0 = P[N_i = 0] = 1 - \lambda\delta,$$
>
> where δ is the length of each slot.

[8] For Bernoulli random variable X, the average is

$$\mathbb{E}X = 0 \times P[X = 0] + 1 \times P[X = 1] = P[X = 1].$$

For conciseness, we usually let $p_0 = P[X = 0]$ and $p_1 = P[X = 1]$. Hence, $\mathbb{E}X = p_1$.

1.8. Discrete-time approximation provides a method for `MATLAB` to generate a Poisson process with arrival rate λ. Here are the steps:

(a) First, we fix the length T of the whole simulation. (For example, $T = 20$ hours.)

(b) Then, we divide T into n slots. (For example, $n = 10,000$.)

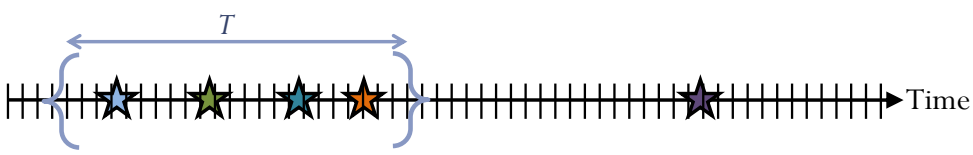
(c) For each slot, only two cases can happen: 1 arrival or no arrival. So, we generate Bernoulli random variable for each slot with $p_1 = \lambda \times T/n$. (For example, if $\lambda = 5$ arrival/hr, then $p_1 = 0.01$.)

To do this for n slots, we can use the command `rand(1,n) < p1` or `binornd(1,p1,1,n)`.

1.9. Note that what we have just generated is exactly ***Bernoulli trials*** whose success probability for each trial is $p_1 = \lambda\delta$. In other words, a Poisson process can be approximated by Bernoulli trials with success probability $p_1 = \lambda\delta$.

1.2 Properties of Poisson Processes

1.10. What we want to do next is to revisit the description of the number of arrivals in a time interval. Now, we will NOT assume that length of the time interval is short. In particular, let's reconsider an interval of length T below.



Let N be the number of arrivals during this time interval. In the picture above, $N = 4$.

Again, we will start with a discrete-time approximation; we divide T into n small slots of length $\delta = \frac{T}{n}$. In the previous subsection, we know that the number of arrivals in these intervals, denoted by $N_1, N_2, \ldots, N_n$ can be well-approximated by i.i.d. Bernoulli with probability of having exactly one arrival $= \lambda\delta$. (Of course, we need δ to be small for the approximation to be

precise.) The total number of arrivals during the original interval of length T can be found by summing the values of the N_i's:

$$(3)$$

You may recall, from introductory probability class, that

(a) summation of n Bernoulli random variables with success probability p gives a binomial(n, p) random variable[9]

and that

(b) a binomial(n, p) random variable whose n is large and p is small can be well approximated by a Poisson random variable with parameter $\alpha = np$ [10].

Therefore, the pmf of the random variable N in (3) can be approximated by a Poisson pmf whose parameter is

$$\alpha = np_1 = n\lambda\frac{T}{n} = \lambda T.$$

This approximation[11] gets more precise when n is large (δ is small). In

[9] X is a ***binomial*** random variable with size $n \in \mathbb{N}$ and parameter $p \in (0, 1)$ if

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \ldots, n\} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

We write $X \sim \mathcal{B}(n, p)$ or $X \sim$ binomial(p). Observe that $\mathcal{B}(1, p)$ is Bernoulli with parameter p. Note also that $\mathbb{E}X = np$. Another important observation is that such random variable is simply a sum of n independent and identically distributed (i.i.d.) Bernoulli random variables with parameter p.

[10] X is a ***Poisson*** random variable with ***parameter*** $\alpha > 0$ if

$$p_X(x) = \begin{cases} e^{-\alpha}\frac{\alpha^x}{x!}, & x \in \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

We write $X \sim \mathcal{P}(\alpha)$ or Poisson(α). Note also that $\mathbb{E}X = \alpha$.

[11] In introductory probability class, you may have seen pointwise convergence in terms of the pmfs. An alternative view on this convergence is to look at the characteristic function. The characteristic function $\varphi_X(u)$ of a random variable X is given by $\mathbb{E}\left[e^{jXu}\right]$. In particular, for Bernoulli random variable with parameter p, the characteristic function is $1 - p + pe^{ju}$. One important property of the characteristic function is that the characteristic function of a sum of independent random variables is simply the product of the characteristic functions of the individual random variables. Hence, the characteristic function of the binomial random variable with parameter pair (n, p) is simply $\left(1 - p + pe^{ju}\right)^n$ because it is a sum of n independent Bernoulli random variables. For us, the parameter p is $\lambda T/n$. So, the characteristic function of the binomial is

$$\left(1 - \frac{1}{n}\left(\lambda T + \lambda T e^{ju}\right)\right)^n.$$

As $n \to \infty$, the binomial characteristic function converges to $\exp\left(-\lambda T + \lambda T e^{ju}\right)$ which turns out to be the characteristic function of the Poisson random variable whose parameter is $\alpha = \lambda T$.

fact, in the limit as $n \to \infty$ (and hence $\delta \to 0$), the random variable N is governed by $\mathcal{P}(\lambda T)$. Recall that the expected value of $\mathcal{P}(\alpha)$ is α. Therefore, λT is the expected value of N. This agrees with what we have discussed before in (2).

In conclusion,

> the number N of arrivals in an interval of length T is a Poisson random variable with mean (parameter) λT

1.11. Now, to sum up what we have learned so far, the following is one of the two main properties of a Poisson process

> The number of arrivals $N_1, N_2, N_3, \ldots$ during non-overlapping time intervals are independent Poisson random variables with the mean being $\lambda\times$ the length of the corresponding interval.

1.12. Another main property of the Poisson process, which provides an alternative method for simulating Poisson process, is that

> The lengths of time between adjacent arrivals $W_1, W_2, W_3, \ldots$ are i.i.d. exponential[12] random variables with mean $1/\lambda$.

The exponential part of this property can be seen easily by considering the complementary cumulative distribution function (CCDF) of the W_k's.

[12]The ***exponential*** distribution is denoted by $\mathcal{E}(\lambda)$. An exponential random variable X is characterized by its probability density function

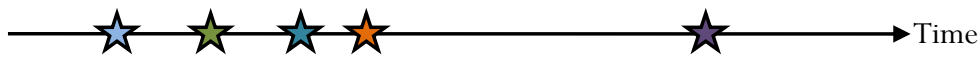
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Note that $\mathbb{E}X = \frac{1}{\lambda}$. The cumulative distribution function (CDF) of X is given by

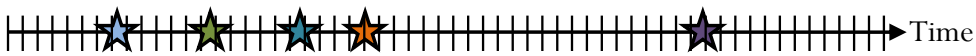
$$F_X(x) \equiv P[X \le x] = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Often, we also talked about the complementary cumulative distribution function (CCDF) which, for exponential random variable, is given by

$$P[X > x] \equiv 1 - F_X(x) = \begin{cases} e^{-\lambda x}, & x > 0, \\ 1, & x \le 0. \end{cases}$$



Alternatively, this property can be derived by looking at the discrete-time approximation of the Poisson process. In the discrete-time version, the time until the next arrival is geometric[13]. In the limit, the geometric random variable becomes exponential random variable.



Both main properties of Poisson process are shown in Figure 2. The small-slot analysis (discrete-time approximation), which can be used to prove the two main properties, is shown in Figure 3.

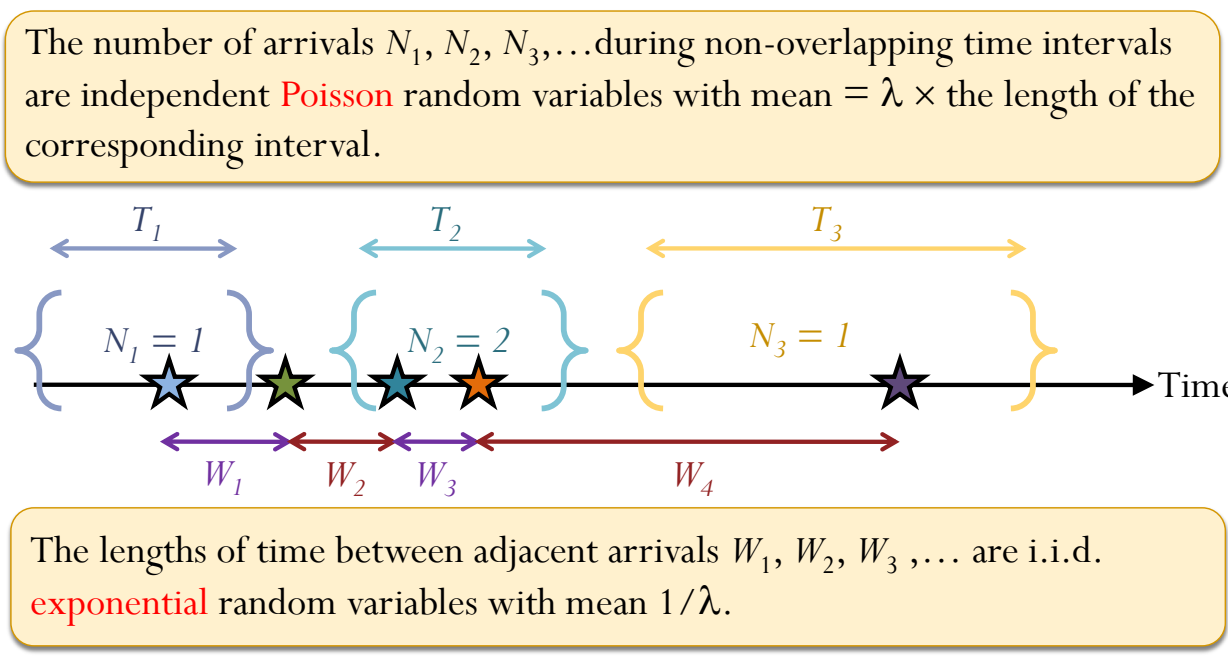


Figure 2: Two main properties of a Poisson process

[13]You may recall from your introductory probability class that, in Bernoulli trials, the number of trials until the next success is geometric.

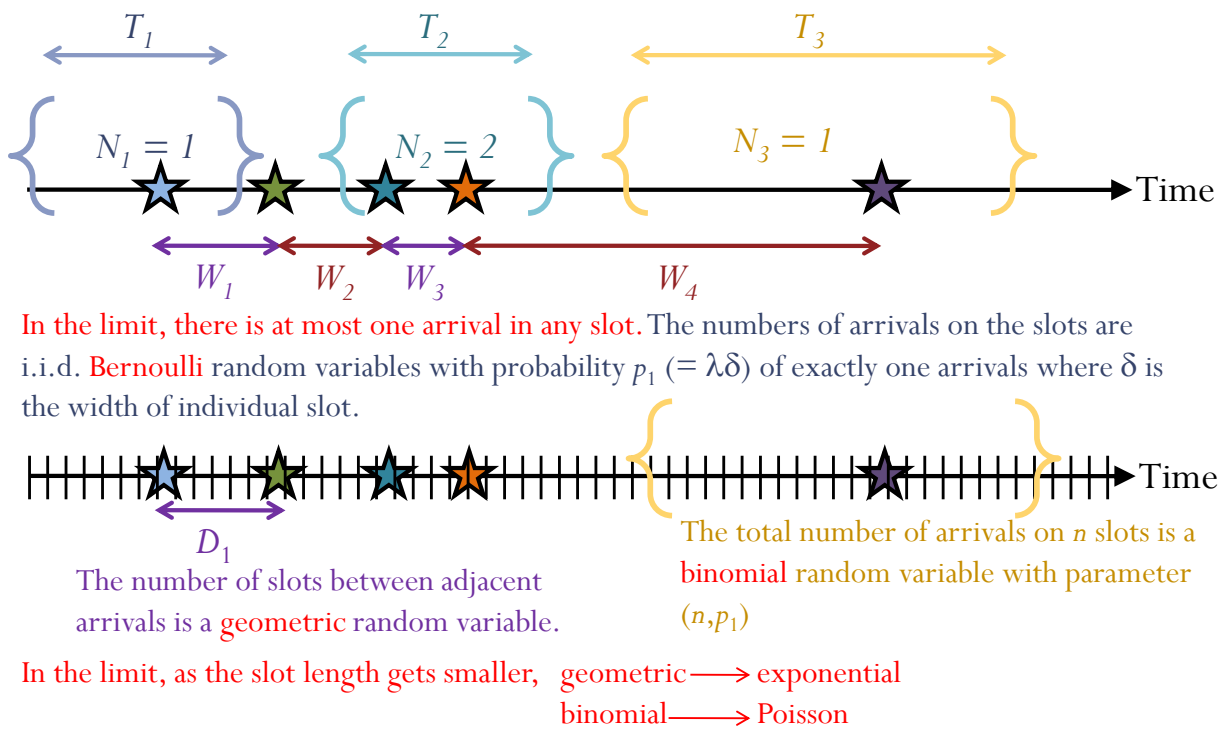


Figure 3: Small slot analysis (discrete-time approximation) of a Poisson process

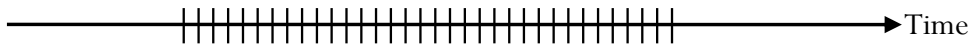
2 Derivation of the Markov chain for Erlang B Formula

In this section, we combine what we know about Poisson process discussed in the previous section with the assumption on the call duration. The goal is to characterize how the random quantity K, eluded to before we begin Section 1, is evolved as a function of time. The key idea is again to use the small-slot (discrete-time) approximation.

2.1. Recall that, for the Erlang B formula, we assume that there are m channels available in the trunking pool. Therefore, the probability P_b that a call requested by a user will be blocked is given by the probability that none of the m channels are free when a call is requested.

We will consider the long-term behavior of this system, i.e. the system is assumed to have been operating for a long time. In which case, at the instant that somebody is trying to make a call, we don't know how many of the channels are currently free.

2.2. Let's first divide the time into small slots (of the same length δ) as we have done in the previous section.



Then, consider any particular slot. Suppose that at the beginning of this time slot, there are K channels that are currently used.[14] We want to find out how this number K changes as we move forward one slot time. This random variable K will be called the ***state*** of the system[15]. The system moves from one state to another one as we advance one time slot.

Example 2.3. Suppose there are 5 persons using the channels at the beginning of a slot. Then, $K = 5$.

(a) Suppose that, by the end of this slot, none of these 5 persons finish their calls.

(b) Suppose also that there is one new person who wants to make a call at some moment of time during this slot.

Then, at the end of this time slot, the number of channels that are occupied becomes

So, the state K of the system changes from 5 to 6 when we reach the end of the slot, which can now be regarded as the beginning of the next slot.

2.4. Our current intermediate goal is to study how the state K evolves from one slot to the next slot. Note that it might be helpful to label the state K as K_1 (or $K[1]$) for the first slot, K_2 (or $K[2]$) for the second slot, and so on.

As shown in Example 2.3, to determine how the K_i's progress from K_1 to K_2, K_2 to K_3, and so on, we need to know two pieces of information:

[14] The value of K can be any integer from 0 to m.

[15] This is the same "state" concept that you have studied in digital circuits class.

Q1 How many calls (that are ongoing at the beginning of the slot under consideration) ***end*** during the slot that we are considering?

Q2 How many ***new*** call requests are made during the slot under consideration?

Note that Q1 depends on the characteristics of the call duration and Q2 depends on the characteristics of the call request/arrival process. After we know the answers to these two questions, then we can find K_i via

$$K_{i+1} = K_i - \underbrace{(\#\text{ old call ends})}_{\text{Q1}} + \underbrace{(\#\text{ new call requests})}_{\text{Q2}}$$

2.5. Q2 is easy to answer.
A2: If the interval is small enough (δ is small), then there can be at most one new arrival (new call request) which occurs with probability

$$p_1 = \lambda\delta.$$

2.6. For Q1, we need to consider the ***call duration model***. The M/M/m/m assumption states that the call duration[16] is exponentially distributed with parameter (rate) μ. Let's consider the call duration D of a particular call.

Recall that the probability density function (pdf) of an exponential random variable X with parameter μ is given by

$$f_X(x) = \begin{cases} \mu e^{-\mu x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

and the average (or expected value) is given by

$$\mathbb{E}[X] = \int_0^\infty x f_X(x) dx = \frac{1}{\mu}.$$

You may remember that in the Erlang B formula, we assume that the average call duration is $\mathbb{E}[D] = H = \frac{1}{\mu}$.

Recall, also, that $P[X > x] = e^{-\lambda x}$ for $x > 0$.

[16] In queueing theory, this is sometimes called the ***service time***. The parameter μ then captures the service rate.

An important property of an exponential random variable X is its memoryless property[17]:

$$P\left[X > x + \delta | X > x\right] = P\left[X > \delta\right].$$

For example,

$$P\left[X > 7 | X > 5\right] = P\left[X > 2\right].$$

What does this memoryless property mean? Suppose you have a lightbulb and you have used it for 5 years already. Let it's lifetime be X. Then, of course, X is a random variable. You know that $X > 5$ because it is still working now. The conditional probability $P\left[X > 7 | X > 5\right]$ is the probability that it will still work after two more years of use (given the fact that currently it has been working for five years). Now, if X is an exponential random variable, then the memoryless property says that $P\left[X > 7 | X > 5\right] = P\left[X > 2\right]$. In other words, the probability that you can use it for at least two more years is exactly the same as the probability that you can use a new lightbulb for more than two years. So, your old lightbulb essentially forgets that it has been used for 5 years, It always performs like a new lightbulb (statistically). This is what we mean by the memoryless property of an exponential random variable.

2.7. To answer Q1, we now return to our small slot approximation. Again, consider one particular slot. At the beginning of our slot, there are $K = k$ ongoing calls. The probability that a particular call, which is still ongoing at the beginning of this slot, will be unfinished by the end of this slot is $e^{-\mu\delta}$.

[17] To see this, first recall the definition of conditional probability: $P\left(A | B\right) = \frac{P(A \cap B)}{P(B)}$. Therefore,

$$P\left[X > x + \delta \,|X > x\right] = \frac{P\left[X > x + \delta \text{ and } X > x\right]}{P\left[X > x\right]} = \frac{P\left[X > x + \delta\right]}{P\left[X > x\right]}.$$

Now, $P\left[X > x\right] = \int\limits_x^\infty \mu e^{-\mu x} dx = e^{-\mu x}$. Hence,

$$P\left[X > x + \delta \,|X > x\right] = \frac{e^{-\mu(x+\delta)}}{e^{-\mu x}} = e^{-\mu\delta} = P\left[X > \delta\right].$$

To see this, consider a particular call. Suppose the duration of this call is D. By assumption, we know that D is exponential with parameter μ.

Let s be the length of time from the call initiation time to the beginning of our slot. Note that the call is still ongoing. Therefore $D > s$. Now, to say that this call will be unfinished by the end of our slot is equivalent to requiring that $D > s + \delta$. By the memoryless property, we have

$$P\left[D > s + \delta \,|D > s\right] = P\left[D > \delta\right] = e^{-\mu\delta}.$$

Recall that we have $K = k$ ongoing calls at the beginning of our slot. So, by the end of our slot, the probability that none of them finishes is

$$\left(e^{-\mu\delta}\right)^k = e^{-k\mu\delta}.$$

The probability that exactly one of them finishes is

Now, note that $e^x \approx 1 + x$ for small x. Therefore, A1:

(a) the probability that none of the $K = k$ calls ends during our slot is

(b) the probability that exactly one of them ends during our slot is

Magically, these two probabilities sum to one. So, we don't have to consider other events/cases.

2.8. Summary:

(a) Call generation/request/initiation process:

(b) Call duration process:

So, after one (small) slot, there can be four events:

#new calls	#old-calls end	Effect on K	Corresponding probability
0	0		
0	1		
1	0		
1	1		

Therefore, if we have $K = k$ at the beginning of our time slot, then, at the end of our slot, "only one" of the following three events can happen:

(a) K remains unchanged with probability $1 - \lambda\delta - k\mu\delta$, or

(b) K decreases by 1 with probability $k\mu\delta$, or

(c) K increases by 1 with probability $\lambda\delta$.

This is illustrated by the diagram in Figure 4:

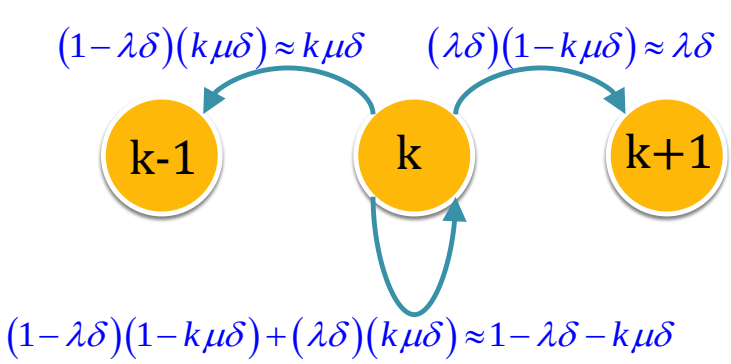


Figure 4: State transition diagram for state k.

Note that the labels on the arrows indicate transition probabilities which are conditional probabilities of going from some value of K to another value.

2.9. When there are m trunked channels, the possible values of K are $0, 1, 2, \ldots, m$. We can combine the diagram above into one diagram that includes all possible values of K:

Note that the arrow $\lambda\delta$ which should go out of state m will return to state m itself because it corresponds to **blocked calls** which do not increase the value of K.

The resulting diagram is called the ***Markov chain state diagram*** for Erlang B.

Example 2.10. Erlang B (Markov chain) state diagram when $m = 2$: